Malaysian Journal of Mathematical Sciences 13(1): 31–40 (2019)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

Fourier-Based Approach for Power Options Valuation

Ibrahim, S. N. I. *[1,2] and Ng, T. W.[1]

[1] *Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, Malaysia*
[2] *Institute for Mathematical Research, Universiti Putra Malaysia, Malaysia*

E-mail: iqmal@upm.edu.my
* *Corresponding author*

Received: 17 January 2018
Accepted: 10 January 2019

ABSTRACT

In this study, we price options whose underlying asset is raised to a constant using the Fourier-Cosine (COS) method. The valuation is made within the Black-Scholes environment, where numerical experiments show that the COS method is more efficient than other known option pricing techniques.

Keywords: Fourier-cosine, option price, power options.

1. Introduction

The most standard option is the vanilla option. At its expiration date, T, for a known underlying share price S and exercise price K, the payoff for an ordinary vanilla call option is given as such:

$$\max\left(S_T - K, 0\right), \tag{1}$$

while for an ordinary vanilla put option is as follows:

$$\max\left(K - S_T, 0\right). \tag{2}$$

Since first introduced, many modifications have been made to the vanilla options to cater the needs of investors. These modifications created exotic options, and one such modification is by taking the underlying share price to a power of a constant value. This option is what known as a power option.

Power option, also known as a leveraged option, has a non-linear payoff of the form:

$$\max\left(S_T^n - K, 0\right), \tag{3}$$

where n is a constant such that n is larger than 1. The analytical pricing formula for a power option is as presented in Ibrahim et al. (2012), Tompkins (2000):

$$PowC_T = S_t^n e^{(n-1)\left(r+\frac{n\sigma^2}{2}\right)(T-t)} N(b_1) - Ke^{-r(T-t)} N(b_2), \tag{4}$$

where:

$$b_1 = \frac{\ln\left(\frac{S_T^n}{K}\right) + n\left(r - \frac{\sigma^2}{2} + n\sigma^2\right)(T-t)}{n\sigma\sqrt{T-t}}, \tag{5}$$

$$b_2 = b_1 - n\sigma\sqrt{T-t}. \tag{6}$$

The advantage of holding a power option is not only because it gives a higher payoff than the standard vanilla option, but it can also be used to hedge non-linear risks Tompkins (2000). Bordag and Mikaelyan (2011) also shows that when trading an underlying within narrow limits, power options can increase the leverage in the market.

The Monte Carlo simulation (MCS) technique Boyle (1977) has been extensively used by many practitioners. However, with every increment of the number of simulations, a larger amount of time is needed to compute a single price of an option. Hence, Carr and Madan (1999) introduces the fast

Fourier transform (FFT) to option pricing. This method has proven to be computationally efficient and can produce a vector of option prices for a vector of the corresponding strikes. This method is widely accepted and has been demonstrated to pricing options under the Black-Scholes environment Ibrahim et al. (2012, 2014), under a mean-reverting process with jumps and stochastic volatility Pillay and O'Hara (2011), under double jumps with stochastic interest rate and volatility Zhang and Wang (2013), and recently under the double exponential jump model with stochastic intensity and volatility Huang et al. (2014).

Then, Fang and Oosterlee (2008) introduces the Fourier-Cosine series expansion (COS) method to option pricing problems which has improved the speed of pricing various types of options, such as vanillas and discrete barrier options Fang and Oosterlee (2009). The COS method has shown to be more computationally efficient than the FFT because its convergence rate is exponential, and does not depend on dampening parameters like the FFT does. Nevertheless, both the FFT and the COS methods require analytical characteristic functions for the underlying asset price process in order to be applied to option pricing problems.

Under the Black-Scholes environment, Ibrahim et al. (2012) implements the FFT algorithm to price power options. This paper aims to implement the COS method to price power options and show the efficiency of the COS method over the FFT and the MCS techniques in pricing power options.

2. Pricing Power Options using the COS Method

Consider the payoff of a power option as given in (3). The price of the power option can be written in integral form as such:

$$PowC_t = Ke^{-r(T-t)} \int_0^\infty \left(e^{s_t} - 1\right) q_T(s) ds_t, \tag{7}$$

where $s_t = \ln\left(\frac{S_T^n}{K}\right)$. Reference Fang and Oosterlee (2008) suggests that some values a and b are chosen so that the following integral can be approximated as follows:

$$\phi_{1,T} = \int_a^b e^{iux} q_T(x) dx \approx \phi_T(u). \tag{8}$$

The expansion of the density function $q_T(s)$ in the interval $[a,b]$ can be approximated by:

$$q_T(s) \quad \approx \quad \frac{A_0}{2} + \sum_{k=1}^{N-1} A_k \cos\left(k\pi \frac{x-a}{b-a}\right), \tag{9}$$

for $s \in [a,b]$ where:

$$A_k = \frac{2}{b-a} \Re\left[\phi_{1,T}\left(\frac{k\pi}{b-a}\right)\exp\left[-ia\left(\frac{k\pi}{b-a}\right)\right]\right]. \tag{10}$$

Replacing Equation (9) into Equation (7) yields the following approximation of the power call option price:

$$\begin{aligned} PowC_t \quad = \quad & Ke^{-r(T-t)}\left\{\frac{A_0}{2}\left[\chi_0(0,b) - \psi_0(0,b)\right]\right. \\ & \left. + \sum_{k=1}^{N-1} A_k \left[\chi_k(0,b) - \psi_k(0,b)\right]\right\}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} \chi_k(c,d) \quad = \quad & \frac{1}{1+\theta^2}\left\{\cos\left[(d-a)\theta\right]e^d - \cos\left[(c-a)\theta\right]e^c\right. \\ & \left.\theta \sin\left[(d-a)\theta\right]e^d - \theta\sin\left[(c-a)\theta\right]e^c\right\}, \end{aligned}$$

and

$$\psi_k(c,d) \quad = \quad \frac{1}{\theta}\left\{\sin\left[(d-a)\theta\right] - \sin\left[(c-a)\theta\right]\right\},$$

for $k \neq 0$, while for $k = 0$,

$$\psi_k(c,d) \quad = \quad d - c,$$

with

$$\theta = \frac{k\pi}{b-a}.$$

Given the risk-neutral probability measure $\mathbf{Q}$, the share price follows an exponential Lévy form as such:

$$S_T^n \quad = \quad S_t^n e^{n\left(r - \frac{\sigma^2}{2}\right)(T-t) + n\sigma W_t}, \tag{12}$$

and the characteristic function of $s_t = \ln S_t^n$ is given as follows:

$$\phi_T(u) \quad = \quad e^{\left[iu\left(s_t + n(r - \frac{1}{2}\sigma^2)(T-t)\right) - \frac{1}{2}n^2\sigma^2u^2(T-t)\right]}. \tag{13}$$

Following Fang and Oosterlee (2008), the truncation range is chosen as such:

$$[a,b] = \left[c_1 - L\sqrt{c_2 + \sqrt{c_4}},\ c_1 + L\sqrt{c_2 + \sqrt{c_4}}\right], \tag{14}$$

where c_n is the n$^{\text{th}}$ cumulant of $\ln\left(\frac{S_T^n}{K}\right)$ given by:

$$\begin{aligned} c_1 &= \ln\left(\frac{S_t^n}{K}\right) + n\left(r - \frac{\sigma^2}{2}\right)(T-t), \\ c_2 &= n^2\sigma^2(T-t), \\ c_4 &= 0. \end{aligned}$$

3. Numerical Experiments

In this section, we present the numerical results. We price the power option using the analytical solution (1), the Monte Carlo simulation (MCS), the fast Fourier transform (FFT) method and the Fourier-Cosine expansion (COS) method. We compare the computational speed and accuracy among these methods. In order to understand whether the COS method is good or not, we calculate the percentage relative error, taken relative to the COS method.

Let $n = 2$. Table 1 documents the power option prices and computational times using the analytical solution, across a range of strike prices $K = \{5, 6, 7, 8, 9, 10\}$ for a given $S = 3$, $r = 0.03$, $\sigma = 0.25$, and $T = 1$. For the MCS, we implement Euler discretization (MCSE) and the Milstein scheme (MCSM), and use $N = 100,000$ simulations with 100 time steps. The prices and computational times obtained via MCSE and MCSM are documented in Tables 2 and 3, respectively.

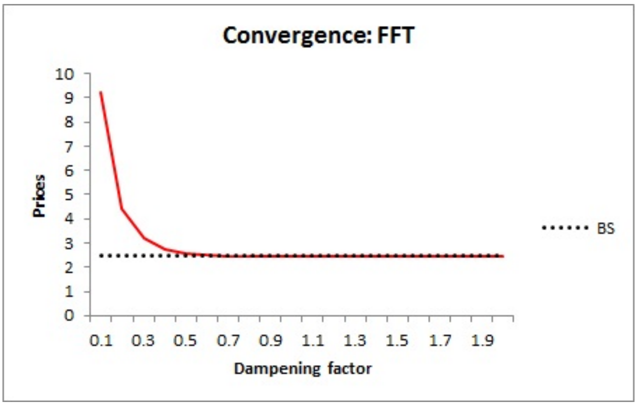

Figure 1: FFT Convergence: α.

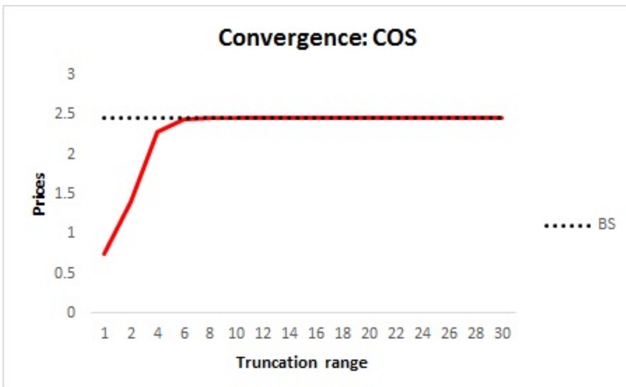

Figure 2: COS Convergence: L.

Moreover, the power option is priced via two Fourier-based methods–the FFT and the COS, where we use $N = 2^{10}$. Figure 1 shows the convergence of the FFT price with respect to the dampening factor α, while Figure 2 shows the convergence of the COS price with respect to the truncation range L. Therefore, we take $\alpha = 3$, and $\delta = 0.7$ for the FFT approach, and $L = 30$ for the COS method. Recall that the Fourier-based approaches produce a range of option prices for a range of strike prices. Hence, we apply a simple linear interpolation to produce a single price for the corresponding strike price. We document the prices and computational times obtained via the FFT and the COS techniques in Tables 4 and 5, respectively.

The analytical solution, the MCSE and the MCSM produce a single price for the power option. On average, the analytical solution produces the prices in 0.0228 seconds (see Table 1), whereas MCSE takes 52.9311 seconds (see Table 2) and MCSM takes 68.5561 seconds (see Table 3). On the other hand, the FFT approach takes on average 0.0367 seconds (see Table 4), while the COS method takes 0.0018 seconds (see Table 5). Even if the FFT takes a longer time than the analytical solution, the FFT and the COS methods produce a range of option prices for a range of strike prices.

Therefore, it shows that the COS method is more efficient in pricing power options than the rests of the option pricing techniques that we use in this study. Furthermore, using the MCS as the benchmark, the COS method produces accurate approximation of the power option prices because the percentage relative error is less than 1% for each of the prices corresponding to each strike price. This is shown in Table 6. Figure 3 plot the prices obtained using all the methods in this study.

Figure 3: Power Option Prices via Analytical Solution, MCS-Euler, MCS-Milstein, FFT and COS.

Table 1: Power Option Prices using Analytical Solution

Strike, K	Prices	Time (sec)
5	5.1386	0.0294
6	4.3275	0.0222
7	3.6088	0.0217
8	2.9871	0.0221
9	2.4593	0.0182
10	2.0173	0.0230

Table 2: Power Option Prices using MCSE

Strike, K	Prices	Time (sec)
5	5.1772	60.0176
6	4.3081	49.8158
7	3.5867	49.8673
8	2.9880	49.4707
9	2.4440	50.9218
10	2.0008	57.4937

Table 3: Power Option Prices using MCSM

Strike, K	Prices	Time (sec)
5	5.1220	75.6438
6	4.3033	71.1997
7	3.5816	66.8861
8	2.9764	68.6406
9	2.4426	64.5755
10	2.0234	64.3907

Table 4: Power Option Prices using FFT

Strike, K	Prices	Time (sec)
5	5.1386	0.0407
6	4.3275	0.0322
7	3.6088	0.0377
8	2.9871	0.0322
9	2.4593	0.0386
10	2.0174	0.0388

Table 5: Power Option Prices using COS

Strike, K	Prices	Time (sec)
5	5.1386	0.0019
6	4.3275	0.0016
7	3.6088	0.0019
8	2.9871	0.0017
9	2.4593	0.0019
10	2.0173	0.0019

Table 6: Percentage Relative Error (%) : COS vs MCS

Strike, K	ϵ, $(MCSE)$	ϵ, $(MCSM)$
5	0.7512	0.3230
6	0.4483	0.5592
7	0.6124	0.7537
8	0.0301	0.3582
9	0.6221	0.6791
10	0.8179	0.6791

4. Conclusion

In this paper, we have applied the COS method to price power options, and compare its accuracy and efficiency with other well-known methods, which are the Monte Carlo simulation and its competitor, the FFT method. We also price the power option using its analytical pricing solution. Despite the non-linearity of the payoff of a power option, our numerical experiments show that the COS method is computationally more efficient than the FFT, and produces more accurate price approximations than the Monte Carlo simulation, either via Euler discretization or the Milstein scheme, for the power options.

Acknowledgement

The authors like to thank the anonymous referees for their valuable feedback, and the authors greatly acknowledge the support from Universiti Putra Malaysia (Putra Grant) with project number GP/2017/9570100 and Ministry of Education Malaysia through Fundamental Research Grant Scheme (02-01-14-1464FR).

References

Bordag, L. A. and Mikaelyan, A. (2011). Models of self-financing hedging strategies in illiquid markets: Symmetry reductions and exact solutions. *Letters in Mathematical Physics*, 96:191–207.

Boyle, P. P. (1977). Options: A Monte Carlo approach. *Journal of Financial Economics*, 4(3):323–338.

Carr, P. and Madan, D. (1999). Option valuation using the fast Fourier transform. *Journal of Computational Finance*, 2(4):61–73.

Fang, F. and Oosterlee, C. (2008). A novel pricing method for european options based on fourier-cosine series expansions. *SIAM Journal on Scientific Computing*, 31(2):826–848.

Fang, F. and Oosterlee, C. (2009). Pricing early-exercise and discrete barrier options by fourier-cosine series expansions. *Numerische Mathematik*, 114(1):27–62.

Huang, J., Zhu, W., and Ruan, X. (2014). Option pricing using the fast fourier transform under the double exponential jump model with stochastic volatility and stochastic intensity. *Journal of Computational and Applied Mathematics*, 263:152–159.

Ibrahim, S. N. I., O'Hara, J. G., and Constantinou, N. (2012). Power option pricing via fast fourier transform. In *2012* 4^{th} *Computer Science and Electronic Engineering Conference (CEEC)*, pages 1–6. IEEE.

Ibrahim, S. N. I., O'Hara, J. G., and Constantinou, N. (2014). Pricing extendible options using the fast fourier transform. *Mathematical Problems in Engineering*, 2014:1–7.

Pillay, E. and O'Hara, J. G. (2011). FFT based option pricing under a mean reverting process with stochastic volatility and jumps. *Journal of Computational Applied Mathematics*, 235(12):3378–3384.

Tompkins, R. (2000). Power options: Hedging nonlinear risks. *Journal of Risk*, 2(2):29–45.

Zhang, S. and Wang, L. (2013). Fast fourier transform option pricing with stochastic interest rate, stochastic volatility and double jumps. *Applied Mathematics and Computation*, 219(23):10928–10933.